Radio Science, Volume 8, Numbers 8, 9, pages 797–804, August-September 1973

A time-domain integral-equation solution for linear antennas and scatterers

Tom K. Liu[1]

Dikewood Corporation, Westwood Research Branch, Los Angeles, California 90024

Kenneth K. Mei

Department of Electrical Engineering and Computer Science and
the Electronics Research Laboratory, University of California at Berkeley,
Berkeley, California 94720

(Received January 11, 1973; revised May 11, 1973.)

The integral equation of Hallen's type is derived in the time domain. The solution of the integral equation is carried out numerically, and the electromagnetic behavior of linear antennas and scatterers under various excitations is presented. The integral equation may be used to obtain time-domain responses of coupled parallel linear antennas, and scatterers with loads. Combining with the method of characteristics in solving transmission line problems, the integral-equation formulation is applied to the cases in which antennas and scatterers are connected to transmission lines.

INTRODUCTION

Recent development in computational techniques has prompted an increasing interest in the direct time-domain investigation of the transient responses of antennas and scatterers. Indeed the time-domain formulation of electromagnetic problems is more fundamental and often offers more physical insight than the familiar steady state approach, and sometimes it also provides computational advantages. Transient behavior of linear antennas has been investigated in terms of superposition of steady state solutions [*Schmitt et al.*, 1966; *Tesche*, 1971] and by space-time-domain integro-differential equations [*Bennett and Auckenthaler*, 1971; *Sayre and Harrington*, 1972; *Miller et al.*, 1973].

Quite recently, a new technique called the singularity expansion method [*Tesche* 1973] has been introduced. This method describes the properties of the wire by the location of the singularities of the response function in the complex frequency plane. Once these locations and the corresponding residues are known, the time responses for any change in the angle of incidence or the time behavior of the incident wave can be easily computed. In this paper, we shall present an alternative approach which results in a Hallen's type space-time domain integral equation; i.e., there are no space-time derivatives of the unknown in the integral operator. As will be outlined in the following paragraph, this method has definite advantages over other methods.

The superposition method is time-consuming computationally: for a simple antenna, it takes over 400 sec on a CDC 6600 computer to calculate the time response of the current on the antenna; whereas the present method takes less than 10 sec on a CDC 6400 computer, which is a considerably slower machine. The superposition method also suffers inaccuracy due to the convergence problem of superimposing many solutions. The singularity expansion method is not very satisfactory in the evaluation of the very early time behavior, and typically it takes about 250 sec on a CDC 6600 computer to search for the singularities and construct the solution. In the absence of information on computational time of the space-time domain integro-differential equation methods, it suffices to point out that the integro-differential equations contain time and/or space derivatives, and the numerical representations of these derivatives are not satisfactory; these methods, as presented, involve complicated numerical procedures and hence are computationally less efficient. Some-

---

[1] Formerly with the Electronics Research Laboratory, University of California at Berkeley, Berkeley, California 94720.

Copyright © 1973 by the American Geophysical Union.

times, the solutions contain wiggles [*Sayre and Harrington,* 1971], indicating an unsatisfactory numerical scheme. Due to the absence of space-time derivatives, the numerical procedure in solving the integral equation is considerably faster than those offered by previous investigators, and we can then afford to compute complicated configurations of antenna arrays.

In a recent paper, *Poggio* [1971] has presented the space-time domain integral equation by the Fourier transform of the familiar frequency domain integral equation. We shall present the derivation of the time-domain equation for linear antennas directly from the space-time potential functions, which appears to be very fundamental in the integral-equation formulation of Maxwell's equations. The mathematical steps of deriving the space-time integral equation also adds new physical insight to the understanding of linear antennas.

The numerical procedures of solving the space-time integral equation is presented and some interesting results are given. All indications show that the space-time integral-equation approach offers accurate results and some computational advantages. The method is then extended to solve for problems involving coupled parallel antennas. Finally, we combine this method with the method of characteristics in solving transmission-line problems by *Branin* [1967] and *Liu* [1968] to deal with the case of a thin wire which is driven by means of a transmission line and the problem of parallel antennas which are connected by transmission lines such as in the case of log-periodic dipole antennas.

MATHEMATICAL FORMULATION

The derivation of the space-time domain integral equation proceeds in parallel to that of Hallen's integral equation in the space-frequency domain. Let us consider a z-oriented wire and start with the wave equation [*Stratton,* 1941] relating the z component of the magnetic vector potential $A_z(z, t)$ and the z-directed incident electric field $E_{\mathrm{inc}}(z, t)$,

$$[\partial^2 A_z(z, t)]/\partial z^2 - c^{-2}[\partial^2 A_z(z, t)]/\partial t^2 = -\epsilon[\partial E_{\mathrm{inc}}(z, t)]/\partial t \qquad (1)$$

where c and ϵ are the velocity of light and the dielectric constant of the medium, respectively. Equation 1 is valid on the surface of a perfect conductor.

A particular solution of equation 1 can be obtained by using the new variables,

$$\eta = ct - z$$

$$\xi = ct + z$$

In terms of ξ and η, equation 1 becomes,

$$[\partial^2/\partial\xi\,\partial\eta)]A_z(\xi, \eta) = (1/4)\epsilon \cdot [(\partial E_{\mathrm{inc}})/\partial t][(\xi - \eta)/2, (\xi + \eta)/2] \qquad (2)$$

The particular solution of equation 2 is,

$$A_z^p(\xi, \eta) = (1/4)\int_{-\infty}^{\eta}\int_{-\infty}^{\xi} \epsilon \cdot [(\partial E_{\mathrm{inc}})/\partial t'][(\xi' - \eta')/2, (\xi' + \eta')/2]\, d\xi'\, d\eta' \qquad (3)$$

Changing the variables back to the (z, t) coordinates we get

$$A_z^p(z, t) = (1/4)\iint_{(z,t)} \epsilon\{[\partial E_{\mathrm{inc}}(z', t')]/\partial t'\} \cdot \{[\partial(\xi', \eta')]/[\partial(z', t')]\}\, dt'\, dz' \qquad (4)$$

where the Jacobian is given by

$$[\partial(\xi', \eta')]/[\partial(z', t')] = \begin{vmatrix} 1 & -1 \\ c & c \end{vmatrix} = 2c$$

The limit of integration in equation 4 should now be changed to (z, t); this procedure is illustrated in Figure 1. The upper limits in equation 3 give

$$\xi' = \xi$$

$$\eta' = \eta$$

which in the (z, t) coordinates become

$$ct' + z' = ct + z$$

$$ct' - z' = ct - z$$

If we integrate t' first in equation 5 the two limits are given by the straight lines

$$t' = t + (z - z')/c$$

$$t' = t - (z - z')/c$$

The integration in equation 4 now becomes

$$\begin{aligned} A_z^p(z, t) &= (\epsilon c/2)\int_{-\infty}^{z}\int_{-\infty}^{t-(z-z')/c} \frac{\partial E_{\mathrm{inc}}(z', t')}{\partial t'}\, dt'\, dz' \\ &\quad + (\epsilon c/2)\int_{z}^{\infty}\int_{-\infty}^{t+(z-z')/c} \{[\partial E_{\mathrm{inc}}(z', t')]/\partial t'\}\, dt'\, dz' \\ &= (\epsilon c/2)\int_{-\infty}^{z} E_{\mathrm{inc}}[z', t - (z - z')/c]\, dz' \\ &\quad + (\epsilon c/2)\int_{z}^{\infty} E_{\mathrm{inc}}[z', t + (z - z')/c]\, dz' \\ &= (\epsilon c/2)\int_{-\infty}^{\infty} E_{\mathrm{inc}}(z', t - |z - z'|/c)\, dz' \end{aligned} \qquad (5)$$

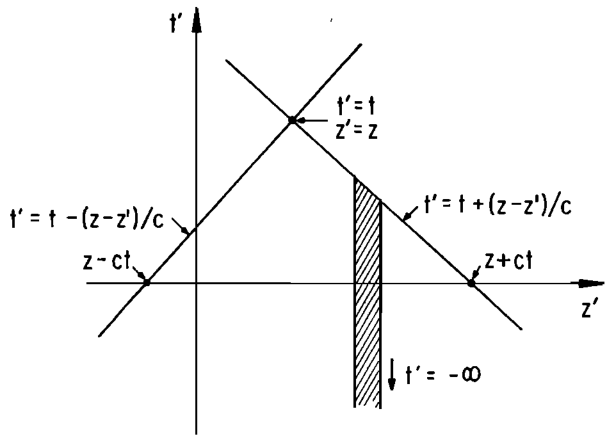

Fig. 1. The limits of integration for A_z^p in (z', t') space.

Since the differential equation (1) is only limited to the surface of the wire, we have

$$A_z^p(z, t) = (2Z_0)^{-1} \int_0^L E_{\rm inc}(z', t - |z - z'|/c)\, dz' \quad (6)$$

where L is the total length of the wire, and $Z_0 = (\epsilon c)^{-1}$ the intrinsic impedance of the medium.

Taking the homogeneous solutions to the wave equation (1), and expressing the vector potential in terms of the z-directed surface-current density $J_z(z, t)$ along the wire [*Stratton,* 1941], we have

$$\iint_S \{[J_z(z', t - |\mathbf{r} - \mathbf{r}'|/c)]/(4\pi |\mathbf{r} - \mathbf{r}'|)\}\, dS' = (2Z_0)^{-1} \int_0^L E_{\rm inc}(z', t - |z - z'|/c)\, dz' + f_1(ct - z) + f_2(ct + z) \quad (7)$$

where S is the surface supporting $J_z(\mathbf{r}, t)$, and $f_1(ct - z)$ and $f_2(ct + z)$ are the homogeneous solutions determined by the boundary conditions that $J_z(0, t) = J_z(L, t) = 0$.

When the observation and source points are far apart, it is possible to use the thin-wire approximation for the kernel in equation 7, with $I(z, t) = 2\pi a\, J_z(z, t)$, where a is the radius of the thin wire, and we obtain

$$\int_0^L [I(z', t - |z - z'|/c)]/\{4\pi[(z - z')^2 + a^2]^{1/2}\}\, dz' = (2Z_0)^{-1} \int_0^L E_{\rm inc}(z', t - |z - z'|/c)\, dz' + f_1(ct - z) + f_2(ct + z) \quad (8)$$

Either equation 7 or 8 can be used for the solution on the thin-wire problems by means of numerical techniques on a digital computer. The physical interpretation of either equation 7 or 8 helps directly towards the procedures of numerical solutions and will be discussed more fully.

THE NUMERICAL METHOD AND RESULTS

Equation 7 or its thin wire version, equation 8, indicates that for a specific pair of values (z, t_0), the current $I(z', t_0 - |z - z'|/c)$ lies on the straight lines

$$t - (z - z')/c = t_0$$

$$t + (z - z')/c = t_0$$

in the (z', t) plane as shown in Figure 2. For a discrete set of parameters t_0, these straight lines form two families of trajectories, α and β, known as the characteristic curves [*Courant and Hilbert,* 1962]. On the α family of curves,

$$t_0 = t + (z - z')/c$$

or

$$dz'/dt = c \quad (9)$$

we find current waves propagating along the wire in the $+z$ direction. Similarly on the β family of curves we find current waves in the $-z$ direction, as

$$dz'/dt = -c \quad (10)$$

The solution of the homogeneous wave equation $f_1(ct - z)$ is invariant along an α curve, which is characterized by a constant value of $ct - z$. Similarly

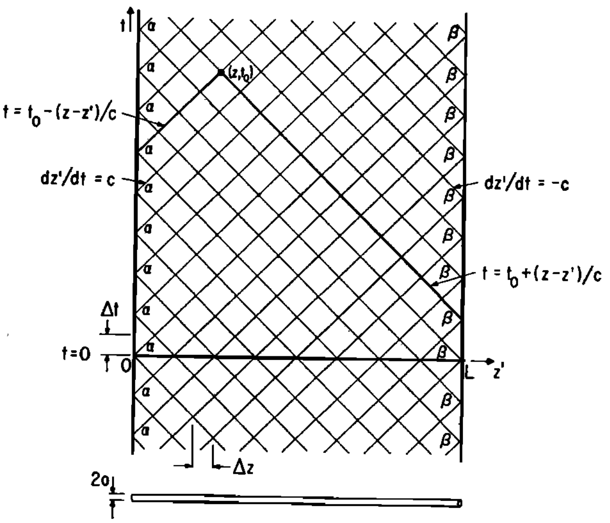

Fig. 2. Diagram of $z' - t$ showing the two families of α and β characteristic curves. Here, $\Delta z = L/16$.

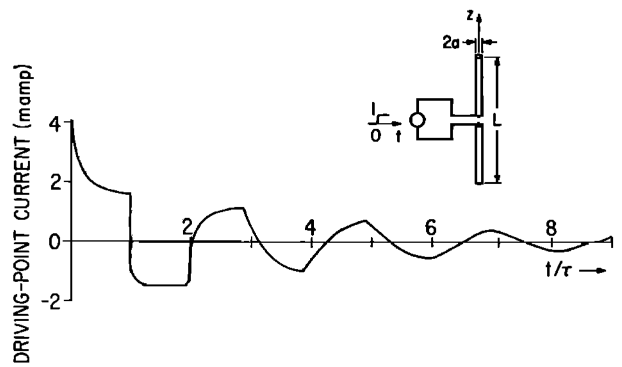

Fig. 3. Driving-point current response for a center-fed antenna under a unit-time step voltage excitation of $\Omega = 2 \log L/a = 10$, where $\tau = L/C$ and t' is the retardation time.

$f_2(ct - z)$ is invariant along a β curve. These two functions should be included in the equations to account for the multiple reflection of the current at the two ends of the wire. Mathematically, the homogeneous solutions enforce the boundary conditions $I(0, t) = I(L, t) = 0$. In effect they summarize the history of the current by the boundary conditions so that the integration may terminate at $z' = 0$ and $z' = L$.

The current at time t may be found from equation 8 if the current and f_1, f_2 along the α and β characteristic curves are known prior to time t. To facilitate the determination of the functions f_1 and f_2, we extend the characteristic curves beyond the $t = 0$ line as shown in Figure 2. Realizing that

$$f_1(ct - z) = f_2(ct + z) = 0 \qquad t \leq 0$$

and that f_1 and f_2 are invariant along these characteristic curves, we find that these functions vanish as long as the (z, t) pair is on a characteristic curve which extends into the region $t \leq 0$. To find the current at $t = \Delta t$, we shall first assume the wire to be unexcited, i.e., $I(z, 0) \equiv 0$, and by the previous reasoning, we find $f_{1,2}(c\Delta t \mp z) = 0$. Thus, the current at $t = \Delta t$ is readily found from equation 8. At $t = 2\Delta t$, since $I(0, t) = I(L, t) = 0$ we find $f_1(2c\Delta t)$ and $f_2(2c\Delta t + L)$ on the outgoing characteristic curves at $z = 0$ and $z = L$, respectively. Using similar steps, we may find $I(z, t)$ at $t = 2\Delta t$, $3\Delta t$, $4\Delta t$, $\cdots$, and $f_1(ct)$ and $f_2(ct + L)$ at $t = 4\Delta t$, $6\Delta t$, $\cdots$. The computation is a step-by-step time-marching process, based on the condition that the currents prior to the point under evaluation are known. Hence, if we can perform the calculation for one time line, we are able to compute for the current at all subsequent time.

The method of the numerical approximation of the integrals in equation 7 or 8 is well-documented [*Harrington,* 1968] and will not be detailed here. To achieve better accuracy, the current is interpolated. As observed from the equations, the current, propagating in the directions of the characteristic curves, should have smooth values along them; hence the interpolation is carried out linearly along the characteristic curves. It is also worth noting that the high frequency components of the solution are considerably improved if the singular integral, i.e., that at the singular cell $z' = z$, is evaluated by the exact current integral with the aid of the method of auxiliary integral [*Lin and Mei,* 1968]. The difference between the so-called exact value and the thin-wire approximation can be more than 10% [*Liu,* 1972]. For nonsingular cells, the thin wire approximation is extremely good for $\Delta z > 5a$.

Some interesting transient responses are presented here. In Figure 3, the driving-point current responses of a center-fed antenna excited by a unit time step voltage is shown for the case $\Omega = 2 \log L/a = 10$. The reflection effects from the two ends, the rapid decay of high-frequency components, and the eventual no-current flow at late time are clearly shown. In Figure 4, the normalized far-field responses of the above antenna are presented for $\theta = 15°$ to $90°$ at $15°$ intervals. The far-field formula

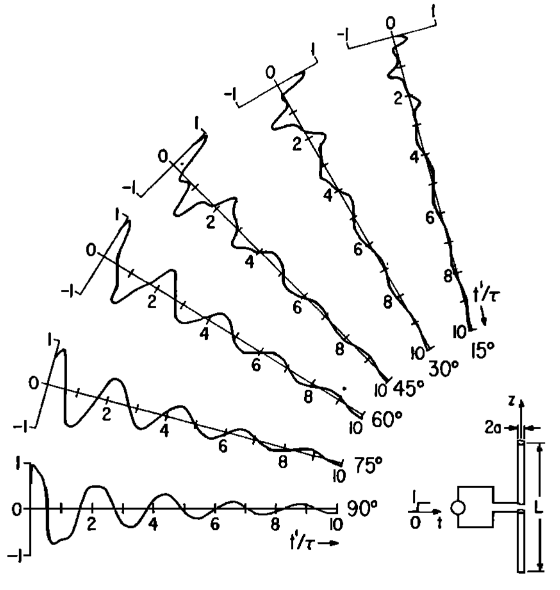

Fig. 4. Normalized far-field responses for a center-fed antenna under a unit-time step voltage excitation of $\Omega = 2 \log L/a = 10$ where $\tau = L/c$ and t' is the retardation time.

is readily derived [*Liu,* 1972] and is given by

$$E_\theta(r, t) = [(\mu \sin \theta)/(4\pi r)](\partial/\partial t) \cdot \int_0^L I[z', t_r + (z'/c) \cos \theta]\, dz' \qquad (11)$$

where t_r is the retarded time to position r.

When the wire acts as a scatterer with a lumped load at $z = z_0$, the electric-field integral in equation 8 has to be modified by introducing an additional term, $-V_{\text{load}}(t - |z - z_0|/c)$, to account for the voltage drop across the load. If the load is an inductance or a capacitance, $V_{\text{load}}(t)$ contains the time derivative or the time integral of the load current, respectively. The problem is then solved by numerical approximation of the derivative or the integral. In Figure 5, the load currents for a center-loaded scatterer under the excitation of a 30° incident plane wave are presented. The resistive load reduces the current amplitude of the unloaded wire. The inductive load increases the time between two subsequent zero crossings of the current, indicating an effective lengthening of the wire; the waveform is also smoothed. The capacitive load shows the opposite effect and effectively shortens the wire; the waveform is sharpened.

It is worth mentioning that by Fourier transforming the transient driving-point current and the applied time-varying voltage, the input impedance of the antenna can be obtained as a function of frequency. The results so obtained agree very well with direct frequency-domain solutions. The time-domain approach, however, has a great saving in computation time over the frequency-domain approach [*Liu,* 1972].

COUPLED PARALLEL ANTENNAS

The above integral equation formulation can easily be extended to the case of coupled parallel antennas. For the case of N antennas, we have

$$\int_{L_j} [I_j(z', t - |z - z'|/c)]/\{4\pi[(z - z')^2 + a_j^2]^{1/2}\}\, dz' + \sum_{\substack{i=1 \\ i \neq j}}^{N} \int_{L_i} \frac{I_i\{z', t - [(z - z')^2 + b_{ij}^2]^{1/2}/c\}}{4\pi[(z - z')^2 + b_{ij}^2]^{1/2}}\, dz'$$

$$= (2Z_0)^{-1} \int_{L_j} E_{\text{inc}}(z', t - |z - z'|/c)\, dz' + f_{1j}(ct - z) + f_{2j}(ct + z) \quad j = 1, \cdots, N \qquad (12)$$

where the subscript j denotes the wire on which z is located, b_{ij} is the perpendicular separation between wires i and j, and a_j is the radius of wire j.

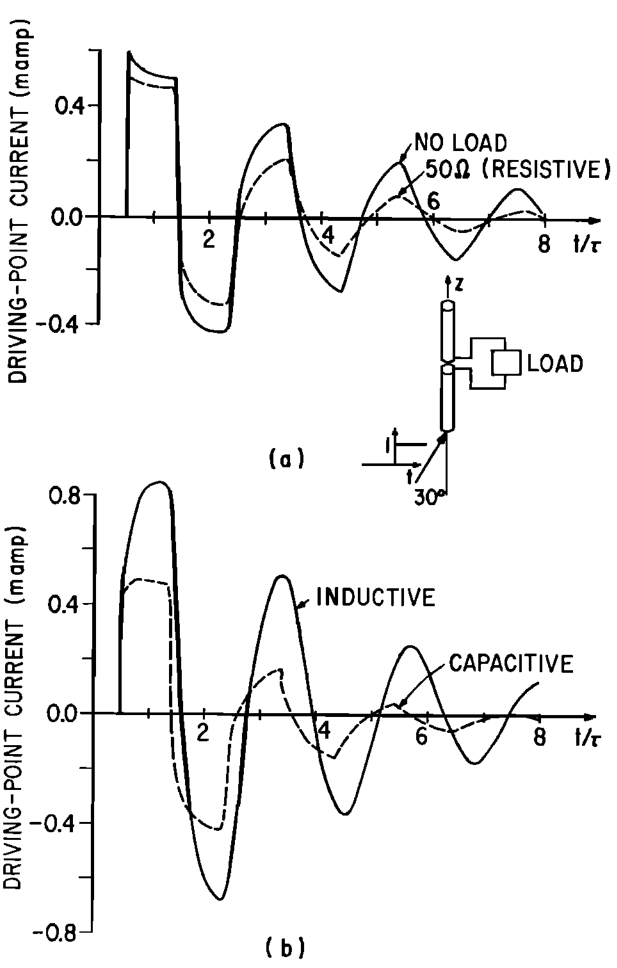

Fig. 5. Induced-load current responses on a center-loaded scatterer under excitation of a 30° incident unit-step electric field: (a) no load and 50 Ω resistive load and (b) inductive load of j 63 Ω and capacitive load of $-j$ 53 Ω at frequency equals c/L. Here, $\Omega = 10$, and $\tau = L/c$.

Equation 12 is similar to equation 8 except for the integrals under the summation sign which account for the coupling effect of the neighboring antennas. The numerical solution procedure follows closely that of a single wire. At time t and at each point z on wire j, there is only one unknown, i.e., $I_j(z, t)$, because the extra integrals contain quantities that are time retarded due to the separation between the antennas i and j, and are known for this step-by-step time-marching process.

To facilitate comparison with the frequency-domain results of *Chang and King* [1968] a pair of nonstaggered, identical parallel antennas is

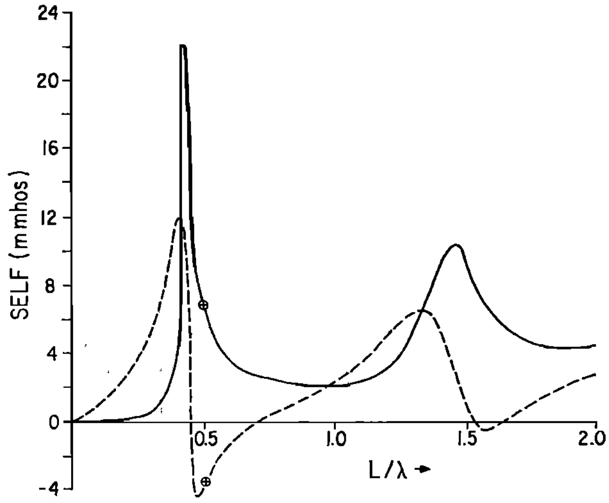

Fig. 6. Self-admittance of a pair of nonstaggered, identical, coupled parallel antennas separated by half the antenna length. Here $\Omega = 8.5$. Legend: —— conductance; – – – susceptance; ⊕ *Chang and King* [1968].

chosen. By applying a voltage to antenna 1 while short-circuiting antenna 2 at the driving point, self- and mutual-admittances can be obtained by Fourier transforming the transient currents and voltage. In Figures 6 and 7 the self- and mutual-admittances of two antennas with $\Omega = 8.5$, separated at a distance equal to half the length of the antennas are presented as a function of frequency. More extensive comparison is given by *Liu* [1972].

WIRES CONNECTED WITH TRANSMISSION LINES

In practice, most antennas are driven by means of transmission lines. We present here a numerical procedure in solving such a problem. Essentially, the method combines two techniques: the numerical method we just outlined for the thin wire and the method of characteristics in solving transmission-line problems as detailed by *Branin* [1967] and *Liu* [1968].

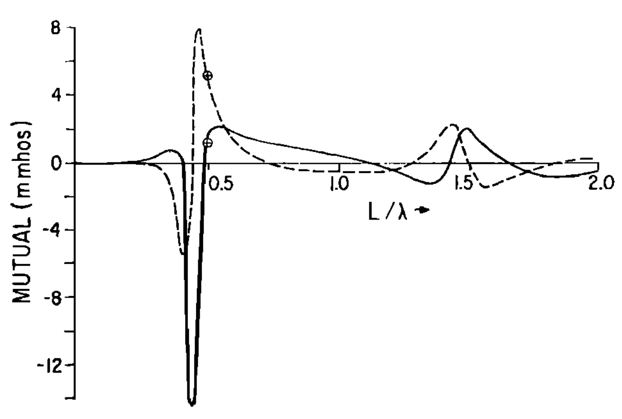

Fig. 7. Mutual admittance of a pair of nonstaggered, identical, coupled parallel antennas separated by half the antenna length. Here $\Omega = 8.5$. Legend: —— conductance; – – – susceptance; ⊕ *Chang and King* [1968].

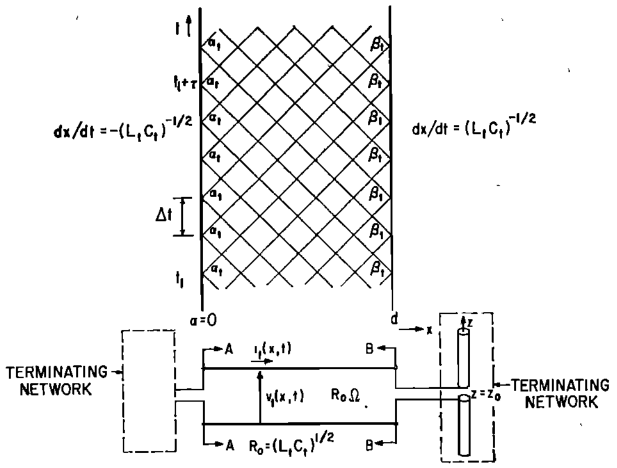

Fig. 8. Diagram of $x - t$ plane showing the two families of α_t and β_t characteristic curve.

In summary, the method of characteristics enables the transmission line at any point to be represented by two Thevenin-equivalent circuits with time-varying sources. Each equivalent circuit is defined along one set of characteristic curves (α_t and β_t as shown in Figure 8). For a lossless transmission line, the equivalent voltage sources,

$$S_\alpha(x, t) = v_t(x, t) + R_0 i_t(x, t)$$

$$S_\beta(x, t) = v_t(x, t) - R_0 i_t(x, t)$$

are invariant along the respective characteristic curves; this property is utilized for the numerical method. For a lossy transmission line, the method is still applicable but is more complicated.

It is important to point out that this method is again a step-by-step time-marching technique and is very compatible with the numerical scheme for the thin wires.

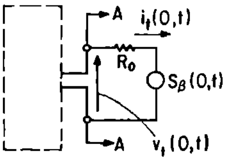

Fig. 9. Diagram of Thevenin-equivalent circuits for $x = 0$ at section A-A in Figure 8. $S_\beta(x, t) = v_t(x, t) - R_0 i_t(x, t)$.

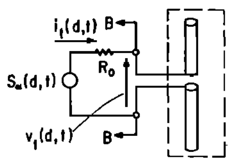

Fig. 10. Diagram of Thevenin-equivalent circuits for $x = d$ at section B-B in Figure 8. $S_\alpha(x, t) = v_t(x, t) - R_0 i_t(x, t)$.

When a transmission line is terminated at $x = d$ by a thin wire at the position $z = z_0$, as shown in Figure 8, the problem, by using the two methods just outlined, reduces to a time-marching solution of the two separate networks in Figures 9 and 10. In this case, we choose equal time steps for the two numerical methods. In Figure 10, the thin wire current at $z = z_0$ being the same as $i_t(d, t)$, can be obtained by solving the following equation which is a modification of equation 8 to take into account the Thevenin equivalent circuit,

$$\int_0^L [I(z', t - |z - z'|/c)]/\{4\pi[(z - z')^2 + a^2]^{1/2}\}\, dz'$$
$$= (2Z_0)^{-1}\left[\int_0^L E_{\mathrm{inc}}(z', t - |z - z'|/c)\, dz' + S_\alpha(d, t - |z - z_0|/c) - R_0 I(z_0, t - |z - z_0|/c)\right]$$
$$+ f_1(ct - z) + f_2(ct + z) \tag{13}$$

The solution procedure is similar to that outlined previously. If it is not a scattering case, the integral on the right-hand side is not required. In Figure 11, we show the result of short-circuit and open-circuit termination at $x = 0$. The scatterer, having $\Omega = 10$, is excited by a unit time step electric field incident at 30° to the z axis. The transmission line has a characteristic impedance of 50 Ω and a transit time τ_t being half of that of the scatterer; it is connected to the midpoint of the wire. We observe the predominantly capacitive behavior for the short-circuit case and inductive behavior for the open-circuit case.

When parallel wires are connected by transmission lines, similar technique is applied. We consider here the case of a pair of identical, nonstaggered parallel antennas with $\Omega = 10$, separated by the length of the antenna, and connected by a 50-Ω transmission line with transit time $\tau_{t2} = \tau$ and driven via another 50-Ω transmission line with transit time $\tau_{t1} = 0.75\ \tau$. In Figure 12, we present the responses when the system is driven by a unit time-step voltage source with a 50-Ω source resistance. We observe in Figure 12 that at large t, when the antenna currents die down, the transmission-line driving-point voltage settles to that of the voltage source, because the load to this transmission line now behaves open-circuited.

The frequency domain results obtained by Fourier transforming the time responses obtained by the above technique compare favorably with those of direct frequency domain methods for both single-wire cases and multiple-wire cases [*Liu*, 1972].

CONCLUSIONS

The formulation of the time-domain Hallen's integral equation is of fundamental importance in the understanding of the transient behavior of linear antennas and scatterers. The equation describes how the current at a space-time coordinate (z, t) depends on previous current values along the wire and the incident electric field. It is pointed out that the current $I(z, t)$ is directly affected by the current that lies on the two characteristic curves passing through (z, t) as well as the previous multiple reflections connected with these characteristic curves. The condition of the reflections is included in the equation by the solutions to the homogeneous wave equation, namely, $f_1(ct - z)$ and $f_2(ct + z)$.

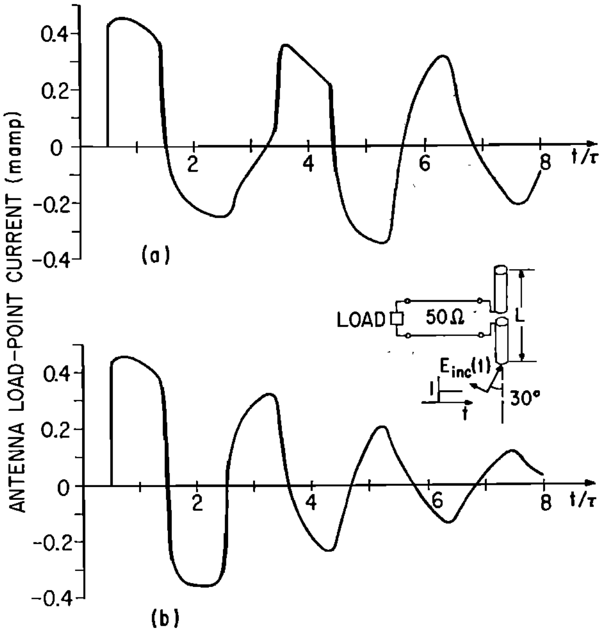

Fig. 11. Load point current of a scatterer with $\Omega = 10$, center-connected to a 50-Ω transmission line (a) with a short-circuit load and (b) with an open-circuit load. Transit time $= \tau/2$, where $\tau = L/c$.

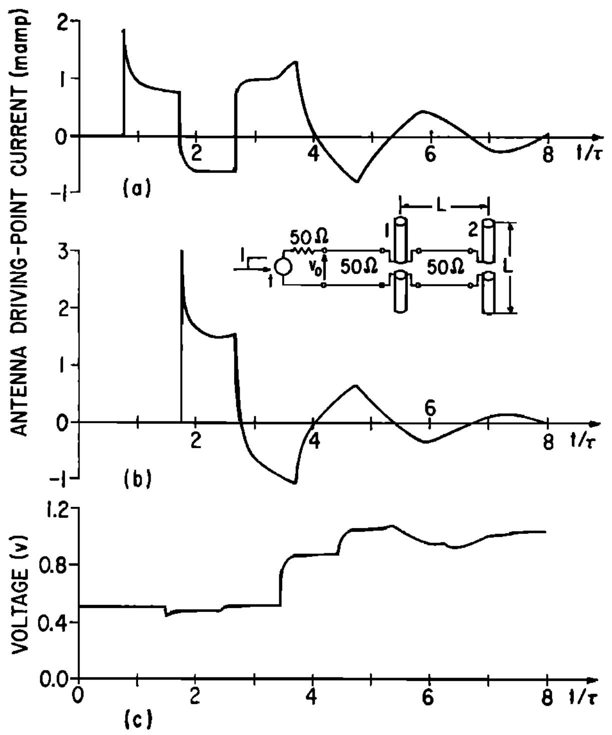

Fig. 12. Antenna driving-point current responses of (a) antenna 1 and (b) antenna 2. (c) The transmission-line driving-point voltage response. For the whole figure $\Omega = 10$, $\tau = L/c$, $\tau_{t1} = 0.75\,\tau$, and $\tau_{t2} = \tau$.

The numerical method is developed with the idea closely related to the physical insight. Interpolation of current along the characteristic curves is used to increase accuracy. The method is relatively simple and can be readily programmed. The solution is fast and accurate and has definite advantages in these respects over the integro-differential equation approaches.

Acknowledgments. The work was supported by National Science Foundation grant GK-24960.

REFERENCES

Bennett, C. L., and A. M. Auckenthaler (1971), Transient and time domain solutions for antennas and scatterers, *IEEE Int. Conv. Dig., 1971,* 624–625.

Branin, F. H., Jr. (1967), Transient analysis of lossless transmission lines, *Proc. IEEE, 55,* 2012–2013.

Chang, V. W. H., and R. W. P. King (1968), On two arbitrarily located identical parallel antennas, *IEEE Trans. Antennas Propagat., AP-16,* 309–317.

Courant, R., and D. Hilbert (1962), *Methods of Mathematical Physics,* vol. 2, pp. 407–550, Interscience, New York.

Harrington, R. F. (1968), *Field Computation by Moment Methods,* pp. 62–81, Macmillan, New York.

Lin, S. H., and K. K. Mei (1968), Numerical solution of dipole radiation in a compressible plasma, *IEEE Trans. Antennas Propagat., AP-16,* 235–241.

Liu, Y. K. (1968), Transient analysis of TEM transmission lines, *Proc. IEEE, 56,* 1090–1092.

Liu, Y. K. (1972), Time domain analysis of linear antennas and scatterers, Ph.D dissertation, pp. 20–82, Department of Electrical Engineering and Computer Sciences, University of California at Berkeley, Berkeley, Calif.

Miller, E. K., A. J. Poggio, and G. J. Burke (1973), An integro-differential equation technique for the time-domain analysis of thin wire structure, 1, The numerical method, *J. Comput. Phys., 12,* 24–48.

Poggio, A. J. (1971), The space-time domain magnetic vector potential integral equation, *IEEE Trans. Antennas Propagat., AP-19,* 702–704.

Sayre, E. P., and R. Harrington (1972), Time domain radiation and scattering by thin wires, *Appl. Sci. Res., 26,* 413–444.

Schmitt, H. S., C. W. Harrison, and C. S. Williams, Jr. (1966), Calculated and experimental response of thin cylindrical antennas to pulse excitation, *IEEE Trans. Antennas Propagat., AP-14,* 120–127.

Stratton, J. A. (1941), *Electromagnetic Theory,* pp. 424–429, McGraw-Hill, New York.

Tesche, F. M. (1971), Numerical analysis of an arbitrarily located antenna within a parallel plate region, *IEEE/G-AP International Symposium, Program and Digest, 1971,* 242–245.

Tesche, F. M. (1973), On the analysis of scattering and antenna problems using the singularity expansion technique, *IEEE Trans. Antennas Propagat., AP-21,* 53–62.